A Completeness Theorem for a 3-Valued Semantics for a First-order Language

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This document presents a Gentzen-style deductive calculus and proves that it is complete with respect to a 3-valued semantics for a language with quantifiers. The semantics resembles the strong Kleene semantics with respect to conjunction, disjunction and negation. The completeness proof for the sentential fragment fills in the details of a proof sketched in Arnon Avron (2003) "Classical Gentzen-type Methods in Propositional Many-valued Logics" in *Beyond Two: Theory and Application of Multiple-Valued Logics*, M. Fitting and E. Orlowska, eds., pp. 117-155. Physica Verlag. The extension to quantifiers is original but uses standard techniques.

Sentential Logic

Let *SL* be a sentential language with connectives $\neg$, $\to$, $\wedge$, $\vee$ and standard syntax. Let *GS3* be a Gentzen-style deductive calculus with the following rules:

(Basis) $A \Rightarrow A$ i.e. $\dfrac{\emptyset}{A \Rightarrow A}$

(Weakening) $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}$

(Cut) $\dfrac{\Gamma_1 \Rightarrow \Delta_1, A \qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$

$(\bot \Rightarrow)$ $\neg A, A \Rightarrow$ i.e. $\dfrac{\emptyset}{\neg A, A \Rightarrow \emptyset}$

$(\neg\neg \Rightarrow)$ $\dfrac{A, \Gamma \Rightarrow \Delta}{\neg\neg\mathrm{A}, \Gamma \Rightarrow \Delta}$

$$(\Rightarrow \neg\neg) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A}$$

$$(\rightarrow\Rightarrow) \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow\rightarrow) \qquad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

$$(\neg \rightarrow\Rightarrow) \qquad \frac{A, \neg B, \Gamma \Rightarrow \Delta}{\neg(A \rightarrow B), \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \neg \rightarrow) \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \rightarrow B)}$$

$$(\wedge \Rightarrow) \qquad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}$$

$$(\Rightarrow \wedge) \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$(\neg\wedge \Rightarrow) \qquad \frac{\Gamma, \neg A \Rightarrow \Delta \qquad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta}$$

$$(\Rightarrow \neg\wedge) \qquad \frac{\Gamma \Rightarrow \Delta, \neg A, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)}$$

$$(\vee \Rightarrow) \qquad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}$$

$$(\Rightarrow \vee) \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$(\neg\vee \Rightarrow) \qquad \frac{\Gamma, \neg A, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta}$$

$$(\Rightarrow \neg\vee) \qquad \frac{\Gamma \Rightarrow, \Delta, \neg A \qquad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \vee B)}$$

NB: Arbitrary rearrangements of elements before "$\Rightarrow$" and arbitrary rearrangements of elements after "$\Rightarrow$" are allowed.

Definition. A *valuation* v is an assignment to atomic sentences of *SL* of members of $\{Y, I, N\}$ ("yes", "indeterminate" and "no").

Definition. V *extends* a valuation v to every sentence of *SL* iff for all sentences P of *SL*:

1. if P is atomic: $V(P) = v(P)$

2. if $P = \neg Q$, then: (a) $V(P) = Y$ if $V(Q) = N$,

   (b) $V(P) = N$ if $V(Q) = Y$,

   (c) $V(P) = I$ otherwise;

3. if $P = (Q \to R)$, then: (a) $V(P) = Y$ if $V(Q) \in \{I, N\}$ or $V(R) = Y$,

   (b) $V(P) = N$ if $V(Q) = Y$ and $V(R) = N$,

   (c) $V(P) = I$ if $V(Q) = Y$ and $V(R) = I$;

4. if $P = (Q \wedge R)$, then: (a) $V(P) = Y$ if $V(Q) = V(R) = Y$,

   (b) $V(P) = N$ if $V(Q) = N$ or $V(R) = N$,

   (c) $V(P) = I$ otherwise;

5. if $P = (Q \vee R)$, then: (a) $V(P) = Y$ if $V(Q) = Y$ or $V(R) = Y$,

   (b) $V(P) = N$ if $V(Q) = V(R) = N$,

   (c) $V(P) = I$ otherwise;

In tables:

$\neg$	
Y	N
I	I
N	Y

$\to$	Y	I	N
Y	Y	I	N
I	Y	Y	Y
N	Y	Y	Y

$\wedge$	Y	I	N
Y	Y	I	N
I	I	I	N
N	N	N	N

$\vee$	Y	I	N
Y	Y	Y	Y
I	Y	I	I
N	Y	I	N

NB: This set of connectives is not functionally complete. That is, not all truth functions on {Y, I, N} can be defined by means of them (Avron 2003, p. 219).

Definition. A *model* for a sequence $\Gamma \Rightarrow \Delta$ is a valuation v s.t. if V extends v, then for some $P \in \Gamma, V(P) \in \{I, N\}$ or for some $P \in \Delta, V(P) = Y$.

Definition. $\dfrac{\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is *valid*$_3$ iff for every valuation v, if v is a model of each of $\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n$, then it is a model of $\Gamma \Rightarrow \Delta$.

Definition. $\Gamma \models_3 \Delta$ iff $\dfrac{\emptyset}{\Gamma \Rightarrow \Delta}$ is valid.

Definition. Where $S = \{\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n\}$, an *S-cut* is an application of (Cut) in which $A \in \left(\bigcup_{i=1}^{n} \Gamma_i\right) \cup \left(\bigcup_{i=1}^{n} \Delta_i\right)$.

Definition. An *S-proof* of $\Gamma \Rightarrow \Delta$ from a set of sequences S is a proof in which every application of (Cut) is an S-cut.

Definition. Where $S = \{\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n\}$, $\Gamma^* \Rightarrow \Delta^*$ is *S-saturated* iff:

1. there is no S-proof of $\Gamma^* \Rightarrow \Delta^*$;
2. if $A \in \left(\bigcup_{i=1}^{n} \Gamma_i\right) \cup \left(\bigcup_{i=1}^{n} \Delta_i\right)$ then $A \in \Gamma^* \cup \Delta^*$;
3. (a) if $\neg\neg A \in \Gamma^*$, then $A \in \Gamma^*$,
   (b) if $\neg\neg A \in \Delta^*$, then $A \in \Delta^*$;
4. (a) if $A \to B \in \Gamma^*$, then $A \in \Delta^*$ or $B \in \Gamma^*$,
   (b) if $A \to B \in \Delta^*$, then $A \in \Gamma^*$ and $B \in \Delta^*$,
   (c) if $\neg(A \to B) \in \Gamma^*$, then $A \in \Gamma^*$ and $\neg B \in \Gamma^*$,
   (d) if $\neg(A \to B) \in \Delta^*$, then $A \in \Delta^*$ or $\neg B \in \Delta^*$;
5. (a) if $A \wedge B \in \Gamma^*$, then $A \in \Gamma^*$ and $B \in \Gamma^*$,
   (b) if $A \wedge B \in \Delta^*$, then $A \in \Delta^*$ or $B \in \Delta^*$,
   (c) if $\neg(A \wedge B) \in \Gamma^*$, then $\neg A \in \Gamma^*$ or $\neg B \in \Gamma^*$,
   (d) if $\neg(A \wedge B) \in \Delta^*$, then $\neg A \in \Delta^*$ and $\neg B \in \Delta^*$;
6. (a) if $A \vee B \in \Gamma^*$, then $A \in \Gamma^*$ or $B \in \Gamma^*$,
   (b) if $A \vee B \in \Delta^*$, then $A \in \Delta^*$ and $B \in \Delta^*$,
   (c) if $\neg(A \vee B) \in \Gamma^*$, then $\neg A \in \Gamma^*$ and $\neg B \in \Gamma^*$,
   (d) if $\neg(A \vee B) \in \Delta^*$, then $\neg A \in \Delta^*$ or $\neg B \in \Delta^*$.

NB: If $\Gamma^* \Rightarrow \Delta^*$ is S-saturated, then membership in Γ^* behaves like Y and membership in Δ^* behaves like I or N.

Let $s_1, s_2, \ldots, s_m, \ldots$ be a list of all formulas that are either subformulas or negations of subformulas in $(\Gamma \cup \Delta) \cup \left(\left(\bigcup_{i=1}^{n} \Gamma_i \right) \cup \left(\bigcup_{i=1}^{n} \Delta_i \right) \right)$.

Construct $\Gamma^* \Rightarrow \Delta^*$ thus:
Let $\Gamma^0 = \Gamma, \Delta^0 = \Delta$.
For all $i \geq 0$, let $\Gamma^{i+1} = \Gamma^i \cup \{s_{i+1}\}, \Delta^{i+1} = \Delta^i$, if there is no S-proof of $\Gamma^i, s_{i+1} \Rightarrow \Delta^i$.
For all $i \geq 0$, let $\Gamma^{i+1} = \Gamma^i, \Delta^{i+1} = \Delta^i \cup \{s_{i+1}\}$, if there is an S-proof of $\Gamma^i, s_{i+1} \Rightarrow \Delta^i$.
Let $\Gamma^* = \bigcup_{i=1}^{\infty} \Gamma^i$ and $\Delta^* = \bigcup_{i=1}^{\infty} \Delta^i$.

Observation 1: $A \in \Gamma^* \cup \Delta^*$ iff A is a subformula or a negation of a subformula of a formula in $(\Gamma \cup \Delta) \cup \left(\left(\bigcup_{i=1}^{n} \Gamma_i \right) \cup \left(\bigcup_{i=1}^{n} \Delta_i \right) \right)$.

Lemma 1. *Suppose there is no S-proof of* $\Gamma \Rightarrow \Delta$. *Then:*
(i) For each $i \geq 0$, $\Gamma^{i+1} \Rightarrow \Delta^{i+1}$ *has no S-proof.*
(ii) There is no S-proof of $\Gamma^* \Rightarrow \Delta^*$.
(iii) Maximality: Let Γ_0 *and* Δ_0 *be sets consisting of formulas that are either subformulas or negations of subformulas of formulas in* $(\Gamma \cup \Delta) \cup \left(\left(\bigcup_{i=1}^{n} \Gamma_i \right) \cup \left(\bigcup_{i=1}^{n} \Delta_i \right) \right)$.
If $\Gamma_0 \nsubseteq \Gamma^*$ *or* $\Delta_0 \nsubseteq \Delta^*$, *then* $\Gamma_0, \Gamma^* \Rightarrow \Delta^*, \Delta_0$ *has an S-proof.*

Proof. (i) By induction:

Basis: By assumption, $\Gamma^0 \Rightarrow \Delta^0$ has no S-proof.

Induction hypothesis: Suppose $\Gamma^i \Rightarrow \Delta^i$ has no S-proof.

Induction step: By the construction, either $\Gamma^{i+1} = \Gamma^i \cup \{s_{i+1}\}$ or $\Delta^{i+1} = \Delta^i \cup \{s_{i+1}\}$. If $\Gamma^{i+1} = \Gamma^i \cup \{s_{i+1}\}$, then, by the construction, $\Gamma^{i+1} \Rightarrow \Delta^{i+1}$ has no S-proof. If $\Delta^{i+1} = \Delta^i \cup \{s_{i+1}\}$, then there is an S-proof of $\Gamma^i, s_{i+1} \Rightarrow \Delta^i$. Suppose $\Gamma^{i+1} \Rightarrow \Delta^{i+1}$, i.e. $\Gamma^i \Rightarrow \Delta^i, s_{i+1}$, has an S-proof. Then by (Cut), $\Gamma^i \Rightarrow \Delta^i$ has an S-proof, contrary to assumption.

(ii) Suppose there is an S-proof of $\Gamma^* \Rightarrow \Delta^*$. Since proofs are finite, there is an i such that $\Gamma_i \Rightarrow \Delta_i$ (in our construction) has an S-proof, contrary to (i).

(iii) Let Γ_0 and Δ_0 be as described. *Case (a)*: $\Gamma_0 \nsubseteq \Gamma^*$. There is $\Gamma_0^{'} \subseteq \Gamma_0$ such that $\Gamma_0^{'} \neq \emptyset$ and $\Gamma_0^{'} \cap \Gamma^* = \emptyset$. By the construction, $\Gamma_0^{'} \subseteq \Delta^*$. So by (Weakening), there is an S-proof of $\Gamma^0, \Gamma^* \Rightarrow \Delta^*, \Delta^0$. *Case (b)*: $\Delta_0 \nsubseteq \Delta^*$. Similarly. $\square$

Lemma 2. *If $\Gamma \Rightarrow \Delta$ has no S-proof, then $\Gamma^* \Rightarrow \Delta^*$ (as in the construction) is S-saturated.*

Proof. Suppose $\Gamma \Rightarrow \Delta$ has no S-proof.

Condition (1) in the definition of S-saturated: By Lemma 1(ii).

Condition (2) in the definition of S-saturated: Suppose, for a reductio, that $A \in \left(\bigcup_{i=1}^{n} \Gamma_i\right) \cup \left(\bigcup_{i=1}^{n} \Delta_i\right)$ but $A \notin \Gamma^* \cup \Delta^*$. By Lemma 1, maximality, $A, \Gamma^* \Rightarrow \Delta^*$ and $\Gamma^* \Rightarrow \Delta^*, A$ have S-proofs. But $\dfrac{\Gamma^* \Rightarrow \Delta^*, A \qquad A, \Gamma^* \Rightarrow \Delta^*}{\Gamma^* \Rightarrow \Delta^*}$ is an application of (Cut) to a member of $\left(\bigcup_{i=1}^{n} \Gamma_i\right) \cup \left(\bigcup_{i=1}^{n} \Delta_i\right)$. So $\Gamma^* \Rightarrow \Delta^*$ has an S-proof, contrary to Lemma 1(ii).

Condition (3): (a) Suppose $\neg\neg A \in \Gamma^*$. $\dfrac{A, \Gamma^* \Rightarrow \Delta^*}{\neg\neg \mathrm{A}, \Gamma^* \Rightarrow \Delta^*}$ is an application of $(\neg\neg \Rightarrow)$. But $\neg\neg A, \Gamma^* = \Gamma^*$. So $\Gamma^* \Rightarrow \Delta^*$ has an S-proof if $A, \Gamma^* \Rightarrow \Delta^*$ has one. So $A, \Gamma^* \Rightarrow \Delta^*$ has no S-proof. So $A \notin \Delta^*$. So by the construction (and Observation 1), $A \in \Gamma^*$. (b) Suppose $\neg\neg A \in \Delta^*$. By $(\Rightarrow \neg\neg)$, $\Gamma^* \Rightarrow \Delta^*$ has an S-proof if $\Gamma^* \Rightarrow \Delta^*, A$ has one. So $\Gamma^* \Rightarrow \Delta^*, A$ has none. So $A \notin \Gamma^*$. So $A \in \Delta^*$.

Condition (4): (a) Suppose $A \to B \in \Gamma^*$. $\dfrac{\Gamma^* \Rightarrow \Delta^*, A \qquad B, \Gamma^* \Rightarrow \Delta^*}{A \to B, \Gamma^* \Rightarrow \Delta^*}$ is an applcation of $(\to\Rightarrow)$. But $A \to B, \Gamma^* = \Gamma^*$. Since $\Gamma^* \Rightarrow \Delta^*$ lacks an S-proof, either (i) $\Gamma^* \Rightarrow \Delta^*, A$ has no S-proof, or (ii) $B, \Gamma^* \Rightarrow \Delta^*$ has no S-proof. Suppose (i). $A \notin \Gamma^*$. So by the construction, $A \in \Delta^*$. Suppose (ii). $B \notin \Delta^*$. So $B \in \Gamma^*$. (b) Suppose $A \to B \in \Delta^*$. $\dfrac{A, \Gamma^* \Rightarrow \Delta^*, B}{\Gamma^* \Rightarrow \Delta^*, A \to B}$ is an application of $(\Rightarrow\to)$. But $\Delta^*, A \to B = \Delta^*$. Since $\Gamma^* \Rightarrow \Delta^*$ has no S-proof $\Gamma^*, A \Rightarrow \Delta^*, B$ has no S-proof. So $A \notin \Delta^*, B \notin \Gamma^*$, which means $A \in \Gamma^*$ and $B \in \Delta^*$. (c) Suppose $\neg(A \to B) \in \Gamma^*$. $\dfrac{A, \neg B, \Gamma^* \Rightarrow \Delta^*}{\neg(A \to B), \Gamma^* \Rightarrow \Delta^*}$ is an application of $(\neg \to\Rightarrow)$. But $\neg(A \to B), \Gamma^* = \Gamma^*$. So since $\Gamma^* \Rightarrow \Delta^*$ has no S-proof, $A, \neg B, \Gamma^* \Rightarrow \Delta^*$ has no S-proof. So $A \notin \Delta^*, \neg B \notin \Delta^*$, so $A \in \Gamma^*, \neg B \in \Gamma^*$. (d) Suppose $\neg(A \to B) \in \Delta^*$. $\dfrac{\Gamma^* \Rightarrow \Delta^*, A \qquad \Gamma^* \Rightarrow \Delta^*, \neg B}{\Gamma^* \Rightarrow \Delta^*, \neg(A \to B)}$ is an application of $(\Rightarrow \neg \to)$. But $\Delta^*, \neg(A \to B) = \Delta^*$. Since $\Gamma^* \Rightarrow \Delta^*$ has no S-proof, either (i) $\Gamma^* \Rightarrow \Delta^*, A$ has no S-proof, or (ii) $\Gamma^* \Rightarrow \Delta^*, \neg B$ has no S-proof. So either $A \notin \Gamma^*$ or $\neg B \notin \Gamma^*$. So either $A \in \Delta^*$ or $\neg B \in \Delta^*$.

Conditions (5) and (6): Similarly. □

Lemma 3. *If $\Gamma^* \Rightarrow \Delta^*$, constructed from $\Gamma \Rightarrow \Delta$, is S-saturated, then there is a valuation that is a model of every sequence in S, but not a model of $\Gamma \Rightarrow \Delta$.*

Proof. Suppose $\Gamma^* \Rightarrow \Delta^*$ is S-saturated. Define valuation v as follows: For all atomic formulas P of SL, $v(P) = \begin{cases} Y\, if\, P \in \Gamma^* \\ I\, if\, P \notin \Gamma^*\, and\, \neg P \notin \Gamma^* \\ N\, if\, \neg P \in \Gamma^* \end{cases}$.

First step: v is well-defined: If $P \in \Gamma$ and $\neg P \in \Gamma^*$, then by (Weakening) $\dfrac{P, \neg P \Rightarrow}{\Gamma^* \Rightarrow \Delta^*}$ will be an S-proof. Since $\Gamma^* \Rightarrow \Delta^*$ (by S-saturation) does not have an S-proof, either $P \notin \Gamma^*$ or $\neg P \notin \Gamma^*$. So the definition of v does not yield both $v(P) = Y$ and $v(P) = N$.

Second step: Suppose V extends v. Prove for all formulas P of SL, if $P \in \Gamma^*$ then $V(P) = Y$ and if $P \in \Delta^*$ then $V(P) \in \{I, N\}$.

By induction:

Basis: The thesis holds for all literals (atomic sentences and negations of atomic sentences): First, consider atomic P. (i) Suppose $P \in \Gamma^*$. $v(P) = V(P) = Y$. (ii) Suppose $P \in \Delta^*$. Since $\Gamma^* \Rightarrow \Delta^*$ has no S-proof, $P \notin \Gamma^*$. If $\neg P \notin \Gamma^*$, then $v(P) = V(P) = I \in \{I, N\}$. If $\neg P \in \Gamma^*$, then $v(P) = V(P) = N \in \{I, N\}$. Next, consider $\neg P$, where P is atomic. (i) Suppose $\neg P \in \Gamma^*$. $v(P) = V(P) = N$. $V(\neg P) = Y$. (ii) Suppose $\neg P \in \Delta^*$. Since $\Gamma^* \Rightarrow \Delta^*$ has no S-proof, $\neg P \notin \Gamma^*$. If $P \notin \Gamma^*$, then $v(P) = V(P) = I$ and $V(\neg P) = I \in \{I, N\}$. If $P \in \Gamma^*$, then $v(P) = V(P) = Y$ and $V(\neg P) = N \in \{I, N\}$.

Induction hypothesis: The thesis holds for $A, B, \neg A$ and $\neg B$.

Induction step: Show that it holds for $\neg\neg A, (A \to B), \neg(A \to B), (A \wedge B), \neg(A \wedge B), (A \vee B)$ and $\neg(A \vee B)$.

$(\neg\neg)$: Suppose $\neg\neg A \in \Gamma^*$. By the definition of S-saturation, $A \in \Gamma^*$. By IH, $V(A) = Y$. $V(\neg\neg A) = Y$. Suppose $\neg\neg A \in \Delta^*$. By the definition of S-saturation, $A \in \Delta^*$. By IH, $V(A) \in \{I, N\}$. $V(\neg\neg A) \in \{I, N\}$.

$(\to)$: Suppose $(A \to B) \in \Gamma^*$. By the definition of S-saturation, $A \in \Delta^*$ or $B \in \Gamma^*$. By IH, $V(A) \in \{I, N\}$, or $V(B) = Y$. $V(A \to B) = Y$. Suppose $(A \to B) \in \Delta^*$. By the definition of S-saturation, $A \in \Gamma^*$ and $B \in \Delta^*$. By IH, $V(A) = Y, V(B) \in \{I, N\}$. $V(A \to B) \in \{I, N\}$.

$(\neg \to)$ Suppose $\neg(A \to B) \in \Gamma^*$. By the definition of S-saturation, $A \in \Gamma^*$ and $\neg B \in \Gamma^*$. By IH, $V(A) = Y$ and $V(\neg B) = Y$, $V(\neg(A \to B)) = Y$. Suppose $\neg(A \to B) \in \Delta^*$. By the definition of S-saturation, $A \in \Delta*$ or $\neg B \in \Delta^*$. By *IH*, $V(A) \in \{I, N\}$ or $V(\neg B) \in \{I, N\}$. $V(A) \in \{I, N\}$ or $V(B) \in \{Y, I\}$. $V(A \to B) \in \{Y, I\}$. $V(\neg(A \to B)) \in \{I, N\}$.

Cases $(\wedge), (\neg\wedge), (\vee), (\neg\vee)$ similarly.

Consequently, v is not a model of $\Gamma^* \Rightarrow \Delta^*$. So since (by Observation 1)

$\Gamma \subseteq \Gamma^*$ and $\Delta \subseteq \Delta^*$, v is not a model of $\Gamma \Rightarrow \Delta$.

Third step: Show that v is a model of every sequence in S. Let $\Gamma_i \Rightarrow \Delta_i$ be an arbitrary member of S. Show that v is a model of $\Gamma_i \Rightarrow \Delta_i$. Suppose, for reductio, that $\Gamma_i \subseteq \Gamma^*$ and $\Delta_i \subseteq \Delta^*$. In that case, by (Weakening), $\Gamma^* \Rightarrow \Delta^*$ has an S-proof, $\dfrac{\Gamma_i \Rightarrow \Delta_i}{\Gamma^* \Rightarrow \Delta^*}$, contrary to Lemma 1(ii). So, either $\Gamma_i \nsubseteq \Gamma^*$ or $\Delta_i \nsubseteq \Delta^*$. But by condition (2) in the definition of S-saturation (or by Observation 1), $\Gamma_i \cup \Delta_i \subseteq \Gamma^* \cup \Delta^*$. So either (i) there is $A \in \Gamma_i$ such that $A \in \Delta^*$, or (ii) there is $A \in \Delta_i$ such that $A \in \Gamma^*$. In case (i) $V(A) \in \{I, N\}$. So v is a model of $\Gamma_i \Rightarrow \Delta_i$. In case (ii) $V(A) = Y$. So v is a model of $\Gamma_i \Rightarrow \Delta_i$. □

Completeness Theorem for GS3: *If* $\dfrac{\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ *is valid*$_3$*, then, where* $S = \{\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n\}$*, there is an S-proof of* $\Gamma \Rightarrow \Delta$ *in GS3.*

Proof. Suppose there is no S-proof of $\Gamma \Rightarrow \Delta$. Then, by Lemma 2, $\Gamma \Rightarrow \Delta$ can be extended to S-saturated $\Gamma^* \Rightarrow \Delta^*$. By Lemma 3,
$\dfrac{\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is not valid$_3$. □

Corollary 1. *If* $\Gamma \models_3 \Delta$ *then there is a proof in GS3 of* $\Gamma \Rightarrow \Delta$.

Extension of these results to *QL*

Suppose that *SL* is now a language like *SL*, defined above, except that the atomic formulas are composed, by the usual syntax, from countably many predicates of each adicity and denumerably many variables and denumerably many individual constants. Let *QL* be a language like *SL* but containing also, for each (individual) variable v, a quantifier $\forall v$. *QL* has the standard syntax, allowing vacuous quantification, and $\exists v$ abbreviates $\neg\forall v\neg$. In any sequence, $\Gamma \Rightarrow \Delta$, the members of $\Gamma \cup \Delta$ are sentences, containing no free variables. Pn/v denotes the result of substituting n for v wherever v occurs free in P. $Pn/v = P$ if and only if v is not in P. A sentence of *QL* is a formula of *QL* containing no free variable.

Let *GQ3* be a Gentzen-style deductive calculus containing all of the rules of *GS3* plus the following:

$(\forall \Rightarrow)$ $\qquad \dfrac{Pn/v, \Gamma \Rightarrow \Delta}{\forall vP, \Gamma \Rightarrow \Delta}$

$(\Rightarrow \forall)$ $\qquad \dfrac{\Gamma \Rightarrow \Delta, Pn/v}{\Gamma \Rightarrow \Delta, \forall vP}$ where n is not in P and not in any member of $\Gamma \cup \Delta$, i.e. n is new, or v is not in P,

$(\neg\forall \Rightarrow)$ $\dfrac{\neg Pn/v, \Gamma \Rightarrow \Delta}{\neg\forall v P, \Gamma \Rightarrow \Delta}$ where n is not in P and not in any member of $\Gamma \cup \Delta$, i.e. n is new, or v is not in P,

$(\Rightarrow \neg\forall)$ $\dfrac{\Gamma \Rightarrow \Delta, \neg Pn/v}{\Gamma \Rightarrow \Delta, \neg\forall v P}$

Define a *3-valued structure* $\mathfrak{M}$ as a triple $\langle U, \Sigma^+, \Sigma^- \rangle$ where U, the universe, is a nonempty set of objects, and for each individual constant n, $\Sigma^+(n) = \Sigma^-(n) = \Sigma(n) \in U$. For each m-ary predicate R, $\Sigma^+(R) \subseteq U^m, \Sigma^-(R) \subseteq U^m$, and $\Sigma^+(R) \cap \Sigma^-(R) = \emptyset$.

Let a *structure and variable assignment* $\mathfrak{M}_g$ be a quadruple $\langle U, \Sigma^+, \Sigma^-, g \rangle$ with U, Σ^+, Σ^- as before and g a partial function over some of the variables of QL such that for each variable v in the range of g: $g(v) \in U$.

$g[v/o]$ is a *variable assigment like* g except that v is in the range of $g[v/o]$, whether or not v was in the range of g, and $g[v/o]$ assigns o to v instead of whatever g assigns to v, if v is in the range of g and does not already assign o to v. $g_\emptyset$ is the *empty variable assignment* with an empty range.

Associate with $\mathfrak{M}_g$ the function h such that for each singular term t of QL that is either an individual constant of QL or a variable of QL in the range of g, $h(t) = \begin{cases} \Sigma(t) \text{ if } t \text{ is an individual constant,} \\ g(t) \text{ if } t \text{ is a variable.} \end{cases}$

A *structure* $\mathfrak{M} = \mathfrak{M}_{g_\emptyset}$.

Associate with each structure and variable assignment $\mathfrak{M}_g$ a function of the same name from formulas of QL into $\{Y, I, N\}$, as follows:

$\mathfrak{M}_g(Rt_1t_2 \ldots t_m) = Y$ iff $\langle h(t_1), h(t_2), \ldots, h(t_m) \rangle \in \Sigma^+(R)$,
$\mathfrak{M}_g(Rt_1t_2 \ldots t_m) = N$ iff $\langle h(t_1), h(t_2), \ldots, h(t_m) \rangle \in \Sigma^-(R)$,
$\mathfrak{M}_g(Rt_1t_2 \ldots t_m) = I$ otherwise,
$\mathfrak{M}_g(\neg P) = Y$ iff $\mathfrak{M}_g(P) = N$,
$\mathfrak{M}_g(\neg P) = N$ iff $\mathfrak{M}_g(P) = Y$,
$\mathfrak{M}_g(\neg P) = I$ otherwise,
$\mathfrak{M}_g((P \to Q)) = Y$ iff $\mathfrak{M}_g(P) \in \{I, N\}$ or $\mathfrak{M}_g(Q) = Y$,
$(\mathfrak{M}_g(P \to Q)) = N$ iff $\mathfrak{M}_g(P) = Y$ and $\mathfrak{M}_g(Q) = N$,
$\mathfrak{M}_g((P \to Q)) = I$ otherwise,
$\mathfrak{M}_g((P \wedge Q)) = \ldots$ as expected,
$\mathfrak{M}_g((P \vee Q)) = \ldots$ as expected,
$\mathfrak{M}_g(\forall v Q) = Y$ iff for all $o \in U$, $\mathfrak{M}_{g[v/o]}(Q) = Y$,
$\mathfrak{M}_g(\forall v Q) = N$ iff for some $o \in U$, $\mathfrak{M}_{g[v/o]}(Q) = N$,

$\mathfrak{M}_g(\forall vQ) = I$ otherwise.

Observation 2: If U is identical to the set of all individual constants of QL and for all individual constants n of QL, $\Sigma(n) = n$, then $\mathfrak{M}(\forall vQ) = Y$ iff for all n of QL, $\mathfrak{M}(Qn/v) = Y$, and $\mathfrak{M}(\forall vQ) = N$ iff for some n of QL, $\mathfrak{M}(Qn/v) = N$.

Observation 3: If v is not in Q, $\mathfrak{M}_g(\forall vQ) = Y$ iff $\mathfrak{M}_g(Q) = Y$ and $\mathfrak{M}_g(\forall vQ) = N$ iff $\mathfrak{M}_g(Q) = N$.

Definition. A structure $\mathfrak{M}$ is a *model* for $\Gamma \Rightarrow \Delta$ iff either there is $P \in \Gamma$ such that $\mathfrak{M}(P) \in \{I, N\}$ or there is $P \in \Delta$ such that $\mathfrak{M}(P) = Y$.

Definition. $\dfrac{\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is *valid*$_{Q3}$ iff every structure that is a model for each of $\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \ldots, \Gamma_n \Rightarrow \Delta_n$ is a model for $\Gamma \Rightarrow \Delta$.

Let C be a denumerable set of individual constants not in QL. QL^+ is QL supplemented by the individual constants in C. By standard techniques, we associate with each formula P of QL^+ having exactly one free variable, two members of C, c_P^+ and c_P^-, called the *witnesses* for P, having the *same birth date*, such that for no formula Q of QL^+ whose witnesses have that same birth date or an earlier birth date does Q contain c_P^+ or c_P^-. c_P^+ is the *positive witness* for P and c_P^- is the *negative witness* for P.

Definition. The *Henkin set* $\mathcal{H}$ for QL^+ is the set of sentences Q of QL^+ such that for each formula P of QL^+ having at most v free, $Q \in \mathcal{H}$ iff

1. n is an individual constant of QL^+ and $Q = (\forall vP \to Pn/v)$ or $Q = (\neg Pn/v \to \neg\forall vP)$, or
2. v is not in P, and $Q = (P \to \forall vP)$ or $Q = (\neg\forall vP \to \neg P)$, or
3. c_P^+ is the positive witness for P and $Q = (Pc_P^+/v \to \forall vP)$, or
4. c_P^- is the negative witness for P and $Q = (\neg\forall vP \to \neg Pc_P^-/v)$.

Lemma 4. *(a)* $\dfrac{(A \to B), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$ *is provable.*

(b) $\dfrac{(A \to B), \Gamma \Rightarrow \Delta}{\Gamma, B \Rightarrow \Delta}$ *is provable.*

Proof. (a)

$$\cfrac{\cfrac{\cfrac{A \Rightarrow A\ (Basis)}{\Gamma, A \Rightarrow \Delta, B, A\ (Weakening)}}{\Gamma \Rightarrow \Delta, A, (A \to B)\ (\Rightarrow\to)} \qquad \cfrac{(A \to B), \Gamma \Rightarrow \Delta}{(A \to B), \Gamma \Rightarrow \Delta, A\ (Weakening)}}{\Gamma \Rightarrow \Delta, A\ (Cut)}$$

(b) Similarly. □

Lemma 5. $\dfrac{\Gamma, (\forall vP \to Pn/v) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$ *is provable.*

$\dfrac{\Gamma, (Pn/v \to \forall vP) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$ *where n is new, or v is not in P, is provable.*

$\dfrac{\Gamma, (\neg\forall vP \to \neg Pn/v) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$ *where n is new, or v is not in P, is provable.*

$\dfrac{\Gamma, (\neg Pn/v \to \neg\forall vP) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$ *is provable.*

Proof. By Lemma 4 and $(\forall \Rightarrow), (\Rightarrow \forall), (\neg\forall \Rightarrow)$ and $(\Rightarrow \neg\forall)$ respectively. For example:

$$\cfrac{\cfrac{\Gamma, (\neg\forall vP \to \neg Pn/v) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\forall vP\ (Lemma\ 4)} \qquad \cfrac{\cfrac{\Gamma, (\neg\forall vP \to \neg Pn/v) \Rightarrow \Delta}{\neg Pn/v, \Gamma \Rightarrow \Delta\ (Lemma\ 4)}}{\neg\forall vP, \Gamma \Rightarrow \Delta\ (\neg\forall \Rightarrow)}}{\Gamma \Rightarrow \Delta\ (Cut)}$$

□

The Elimination Theorem *Suppose every sentence in*

$(\Gamma \cup \Delta) \cup \left(\left(\bigcup_{i=1}^{n} \Gamma_i \right) \cup \left(\bigcup_{i=1}^{n} \Delta_i \right) \right)$ *is in QL. Suppose also that*

$\dfrac{\Gamma_1, \mathcal{H} \Rightarrow \Delta_1, \ldots, \Gamma_n, \mathcal{H} \Rightarrow \Delta_n}{\Gamma, \mathcal{H} \Rightarrow \Delta}$ *is provable in GQ3 for QL^+. Then*

$\dfrac{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ *is provable in GQ3 for QL.*

Proof. Suppose the hypothesis. Since proofs are finite, there is a finite subset $\mathcal{J} \subseteq \mathcal{H}$ such that $\dfrac{\Gamma_1, \mathcal{J} \Rightarrow \Delta_1, \ldots, \Gamma_n, \mathcal{J} \Rightarrow \Delta_n}{\Gamma, \mathcal{J} \Rightarrow \Delta}$ is provable. Show: $\dfrac{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is provable (in *GQ3* for *QL*). By induction on the size of $\mathcal{J}$:

Basis: $\mathcal{J} = \emptyset$. Trivial.

Induction Hypothesis: Suppose the thesis holds when $\mathcal{J}$ hast m members ($m \geq 0$). Show that the thesis holds when $\mathcal{J}$ has $m + 1$ members.

Case 1: At least one member Q of $\mathcal{J}$ is of the form $(\forall vP \to Pn/v)$ or $(\neg Pn/v \to \neg\forall vP)$ or $(P \to \forall vP)$ or $(\neg\forall vP \to \neg P)$. There is a set $\mathcal{J}'$ such that $\mathcal{J} = \mathcal{J}' \cup \{Q\}, Q \notin \mathcal{J}'$. By the IH it suffices to show that if

$\dfrac{\Gamma_1, \mathcal{J}' \cup \{Q\} \Rightarrow \Delta_1, \ldots, \Gamma_n, \mathcal{J}' \cup \{Q\} \Rightarrow \Delta_n}{\Gamma, \mathcal{J}' \cup \{Q\} \Rightarrow \Delta}$ is provable then

$\dfrac{\Gamma_1, \mathcal{J}' \Rightarrow \Delta_1, \ldots, \Gamma_n, \mathcal{J}' \Rightarrow \Delta_n}{\Gamma, \mathcal{J}' \Rightarrow \Delta}$ is provable, thus:

$$\dfrac{\dfrac{\dfrac{\Gamma_1, \mathcal{J}' \Rightarrow \Delta_1}{\Gamma_1, \mathcal{J}' \cup \{Q\} \Rightarrow \Delta_1 \ (Weakening)} \cdots\cdots \dfrac{\Gamma_n, \mathcal{J}' \Rightarrow \Delta_n}{\Gamma_n, \mathcal{J}' \cup \{Q\} \Rightarrow \Delta_n \ (Weakening)}}{\Gamma, \mathcal{J}' \cup \{Q\} \Rightarrow \Delta \ (Supposition)}}{\Gamma, \mathcal{J}' \Rightarrow \Delta \ (by\, Lemma\, 5)}.$$

Case 2: All members of $\mathcal{J}$ are sentences of the form $(Pc_P^+/v \to \forall vP)$ or $(\neg\forall vP \to \neg Pc_P^-/v)$. Of all the witnesses in the sentences in $\mathcal{J}$ (positive or negative), let c_P^* be one that has latest birth date. (There might be two, one positive, one negative.) c_P^* does not occur in any member of $(\Gamma \cup \Delta) \cup \left(\left(\bigcup_{i=1}^{n} \Gamma_i\right) \cup \left(\bigcup_{i=1}^{n} \Delta_i\right)\right)$ or in any other member of $\mathcal{J}$. Let Q be a sentence in $\mathcal{J}$ containing c_P^*. So $Q = (Pc_P^*/v \to \forall vP)$ or $Q = (\neg\forall vP \to \neg Pc_P^*/v)$. As in Case 1, we can drop Q from the proof. □

Conjecture 1. *A* valuation val for QL^+ *is an assignment of the members of* $\{Y, I, N\}$ *to sentences of* QL^* *that are either quantifications or atomic* (i.e., not negations, not conditionals, etc.).

Definition. *Val extends val* to every sentence of QL^+ in accordance with tables given above for *SL*.

The Henkin Construction Theorem: *Suppose val is a valuation for* QL^+ *and Val extends val such that for all* $Q \in \mathcal{H}, Val(Q) = Y$. *Then we can construct a structure* $\mathfrak{M}_{Val}$ *for* QL^+ *such that for all sentences* P *of* QL^+, $Val(P) = Y$ *iff* $\mathfrak{M}_{Val}(P) = Y$, *and* $Val(P) = N$ *iff* $\mathfrak{M}_{Val}(P) = N$ (By implication: $Val(P) = I$ *iff* $\mathfrak{M}_{Val}(P) = I$).

Proof. Define $\mathfrak{M}_{Val}$ thus:

U is identical to the set of individual constants of QL^+.

For each individual constant n of QL^+: $\Sigma(n) = n$.

For each m-place predicate R of $QL^+(QL)$:

$\Sigma^+(R) = \{\langle n_1, n_2, \ldots, n_m\rangle | val(Rn_1n_2 \ldots n_m) = Y\}$,

$\Sigma^-(R) = \{\langle n_1, n_2, \ldots, n_m\rangle | val(Rn_1n_2 \ldots n_m) = N\}$.

By induction on the length of sentences:

Basis: Suppose P is atomic, i.e. $P = Rn_1n_2 \ldots n_m$.

Left-to-right: Suppose $Val(Rn_1n_2 \ldots n_m) = Y$. By the construction of $\mathfrak{M}_{Val}, \langle n_1, n_2, \ldots, n_m\rangle \in \Sigma^+(R)$. By the construction of $\mathfrak{M}_{Val}$, $\langle\Sigma(n_1), \Sigma(n_2),$

$\ldots, \Sigma(n_m)\rangle \in \Sigma^+(R)$. So by the definition of $\mathfrak{M}_g$ (as a function), $\mathfrak{M}_{Val}(Rn_1n_2\ldots n_m)$ $= Y$. Suppose $Val(Rn_1n_2\ldots n_m) = N$. By the construction, $\langle n_1, n_2, \ldots, n_m\rangle \in$ $\Sigma^-(R)$. By the construction, $\langle\Sigma(n_1), \Sigma(n_2), \ \ldots, \Sigma(n_m)\rangle \in \Sigma^+(R)$. So $\mathfrak{M}_{Val}(Rn_1n_2\ldots n_m) = N$.

Right-to-left: Suppose $\mathfrak{M}_{Val}(Rn_1n_2\ldots n_m) = Y$. By the definition of $\mathfrak{M}_g$ (as a function), $\langle\Sigma(n_1), \Sigma(n_2), \ \ldots, \Sigma(n_m)\rangle \in \Sigma^+(R)$. By the construction of $\mathfrak{M}_{Val}$, $\langle n_1, n_2, \ldots, n_m\rangle \in \Sigma^+(R)$. By the construction of $\mathfrak{M}_{Val}$, $Val(Rn_1n_2\ldots n_m) =$ Y. Suppose $\mathfrak{M}_{Val}(Rn_1n_2\ldots n_m) = N$. Similarly.

Induction hypotheses: Suppose the thesis holds for all sentences having complexity k. Show that it holds for all sentences having complexity k+1.

Induction step:

($\neg$): Exercise.

($\rightarrow$): Suppose $P = (Q \rightarrow R)$.

Left-to-right: Suppose $Val(Q \rightarrow R) = Y$. By the definition of *Val*, either $Val(Q) \in \{I, N\}$ or $Val(R) = Y$. By IH, either $\mathfrak{M}_{Val}(Q) \in \{I, N\}$ or $\mathfrak{M}_{Val}(R) =$ Y. By the definition of $\mathfrak{M}_g, \mathfrak{M}_{Val}(Q \rightarrow R) = Y$. Suppose $Val(Q \rightarrow R) = N$. By the definition on *Val*, $Val(Q) = Y$ and $Val(R) = N$. By the definition of $\mathfrak{M}_g, \mathfrak{M}_{Val}(Q \rightarrow R) = N$.

Right-to-left: Exercise.

($\wedge$), ($\vee$): Exercise.

($\forall$): *Left-to-right:* Suppose $Val(\forall vQ) = Y$. By the definition of $\mathcal{H}$, for all individual constants n in $QL^+, Val(\forall vQ \rightarrow Qn/v) = Y$. So by the definition of *Val*, for all individual constants n in QL^+, $Val(Qn/v) = Y$. By the IH, for all individual constants n in QL^+, $\mathfrak{M}_{Val}(Qn/v) = Y$. By Observation 2, $\mathfrak{M}_{Val}(\forall vQ) = Y$. Suppose $Val(\forall vQ) = N$. $Val(\neg\forall vQ) = Y$. *Case 1*: v is not in Q. Then by the construction of $\mathcal{H}$, $Val(\neg\forall vQ \rightarrow \neg Q) = Y$. By the definition of $Val, Val(\neg Q) = Y$. By IH, $\mathfrak{M}_{Val}(\neg Q) = Y$. By the definition of $\mathfrak{M}_g, \mathfrak{M}_{Val}(Q) = N$. By Observation 3, $\mathfrak{M}_{Val}(\forall vQ) = N$. *Case 2*: v is in Q. Then by the construction of $\mathcal{H}, Val(\neg\forall vQ \rightarrow \neg Qc^-_Q/v) = Y$. By the definition of $Val, Val(\neg Qc^-_Q/v) = Y$. By the IH, $\mathfrak{M}_{Val}(\neg Qc^-_Q/v) = Y$. By the definition of $\mathfrak{M}_g, \mathfrak{M}_{Val}(Qc^-_Q/v) = N$. By the definition of $\mathfrak{M}_g, \mathfrak{M}_{Val}(\forall vQ) = N$.

Right-to-left: Suppose $\mathfrak{M}_{Val}(\forall vQ) = Y$. *Case 1:* v is not in Q. By Observation 3, $\mathfrak{M}_{Val}(Q) = Y$. By IH, $Val(Q) = Y$. By the construction of $\mathcal{H}, Val(Q \rightarrow$ $\forall vQ) = Y$. $Val(\forall vQ) = Y$. *Case 2*: v is in Q. By Observation 2, for all individual constants n in $QL^+, \mathfrak{M}_{Val}(Qn/v) = Y$. In particular, $\mathfrak{M}_{Val}(Qc^+_Q/v)$ $= Y$. By IH, $Val(Qc^+_Q/v) = Y$. By the construction of $\mathcal{H}$, $Val(Qc^+_Q/v \rightarrow$ $\forall vQ) = Y$. $Val(\forall vQ) = Y$. Suppose $\mathfrak{M}_{Val}(\forall vQ) = N$. By Observation

2, there is an individual constant n of QL^+ such that $\mathfrak{M}_{Val}(Qn/v) = N$. By IH, $Val(Qn/v) = N$. $Val(\neg Qn/v) = Y$. By the construction of $\mathcal{H}$, $Val(\neg Qn/v \to \neg\forall vP) = Y$. $Val(\neg\forall vP) = Y$. $Val(\forall vP) = N$. □

Completeness Theorem for GQ3: *If* $\dfrac{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ *is valid*$_{Q3}$*, then* $\dfrac{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ *is provable in GQ3.*

Proof. Suppose that $\dfrac{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is not provable in *GQ3*. By the Elimination Theorem, $\dfrac{\Gamma_1, \mathcal{H} \Rightarrow \Delta_1, \ldots, \Gamma_n, \mathcal{H} \Rightarrow \Delta_n}{\Gamma, \mathcal{H} \Rightarrow \Delta}$ is also not provable in *GQ3*. So it is also not provable in *GS3*. By the Completeness Theorem for *GS3*, $\dfrac{\Gamma_1, \mathcal{H} \Rightarrow \Delta_1, \ldots, \Gamma_n, \mathcal{H} \Rightarrow \Delta_n}{\Gamma, \mathcal{H} \Rightarrow \Delta}$ is not valid$_3$. So there is a valuation *val* such that *val* is a model for each of $\Gamma_1, \mathcal{H} \Rightarrow \Delta_1, \ldots, \Gamma_n, \mathcal{H} \Rightarrow \Delta_n$, but not a model for $\Gamma, \mathcal{H} \Rightarrow \Delta$. Since *val* is not a model for $\Gamma, \mathcal{H} \Rightarrow \Delta$, for all $Q \in \mathcal{H}, Val(Q) = Y$. By the Henkin Construction Theorem, there is a structure $\mathfrak{M}_{Val}$, such that for all P of QL^+, $Val(P)$ if and only if $\mathfrak{M}_{Val}(P) = Y$. So $\mathfrak{M}_{Val}$ is a model for each of $\Gamma_1, \Rightarrow \Delta_1, \ldots, \Gamma_n, \Rightarrow \Delta_n$, but not for $\Gamma \Rightarrow \Delta$. So $\dfrac{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is not valid$_{Q3}$. □